Geometrization of the Lax Pair Tensors

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Abstract

The tensorial form of the Lax pair equations are given in a compact and geometrically transparent form in the presence of Cartan's torsion tensor. Three dimensional spacetimes admitting Lax tensors are analyzed in detail. Solutions to Lax tensor equations include interesting examples as separable coordinates and the Toda lattice.

1 Introduction

In a series of papers, Rosquist et all.[1, 2, 3, 4, 5] introduced the Lax tensors and presented some models for which Lax tensors exist. The Lax representation for a given dynamical system is not unique [6]. Consequently, the Lax tensor equations

$$D_\delta L_{\alpha\beta\gamma} + D_\gamma L_{\alpha\beta\delta} = L_{\alpha\mu(\gamma} B^\mu{}_{|\beta|\delta)} - B_{\alpha\mu(\gamma} L^\mu{}_{|\beta|\delta)} \qquad (1)$$

depend on an arbitrary third rank object $B^\alpha{}_{\beta\gamma}$, whose specific form can be found through a suitable geometrization of the system.

The solutions of the Lax tensors were investigated on the "dual" manifold[7] in two dimensions [8]. The connection between Lax tensors and Killing-Yano tensors of order three has also been well-established [1]. Killing-Yano tensors of order three were introduced long time ago by Bocnher [9] and Yano [10] as a natural generalization of a Killing vector. Gibbons et all. [11] found that Killing-Yano tensors can be understood as an object generating a "non-generic symmetry", i.e., a supersymmetry appearing only in the specific space-time.

An alternative way to introduce Lax tensors is to consider the Killing and Killing-Yano tensors of order three. Since the Killing-Yano tensors of order three are the generators of the "non-generic" suppersymmetry on a given manifold [12], investigation of the manifolds admitting Lax tensors becomes interesting.

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Untill now a Killing tensor could only be found by solving the Killing or the Killing-Yano equations [13, 14] of order two, or by calculating the Nijenhuis tensor [15]. The existence of the Lax tensors of order three open a new possibility for finding Killing tensors.

The three-particle open Toda lattice was geometrized by a suitable canonical transformation and it was found that the tensor $B^{\alpha}{}_{\beta\gamma}$ is antisymmetric with respect to its first two indices [2]. It is also known that the geometric duality can be generalized to spinning spaces, at an expense of introducing a torsion on the manifold [7]. All these results suggest that torsion can play an important role in the description of Lax tensors.

There are interesting questions yet to be answered as to the Lax equations, and as to the geometrical interpretations of its constituent tensors. Specifically, the properties of the manifold admitting Lax tensors deserve further investigation. Although, on a very general context answers to these questions are not easy to provide, problems concerning Lax tensor equations seem to be manageable when some symmetry properties are imposed to its constituent tensors, in some particular dimensions, which is what we intend to present here.

This letter is organized as follows: In Sec. 2 the Lax tensor equations are rederived in the presence of torsion to provide a geometric meaning to the arbitrary tensor $B^{\alpha}{}_{\beta\gamma}$. In Sec. 3 the Lax tensor equations are analyzed in three dimensions for various symmetry properties of this tensor. The integrability conditions of these equations are discussed for specific cases such as flat and curved space-times as well as spacetimes having torsion. We present the explicit expressions for the Lax tensors for a number of interesting examples including separable orthogonal systems and the three-particle open Toda lattice. The last section is devoted to our comments and conclusions.

2 The tensorial Lax pair equations in the presence of torsion

In this section we rederive the tensorial Lax pair equations by taking into account a suitable definition for the Dirac-Poisson brackets when torsion is introduced on an n-dimensional manifold. The manifold is endowed with a metric

$$ds^2 = g_{\mu\nu}dq^{\mu}dq^{\nu} \tag{2}$$

and with an affine connection $\hat{\Gamma}^{\lambda}{}_{\mu\nu}$, satisfying the metricity condition, and thereby related to the Christoffel symbols and Cartan's torsion tensor as:

$$\hat{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + T^{\lambda}{}_{\mu\nu}. \tag{3}$$

The torsion tensor is assumed to be completely antisymmetric to fit the autoparallels with the geodesics of the manifold. The Hamiltonian for a dynamical

system is constructed as

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \tag{4}$$

On the phase space, the expression for the covariant derivative with torsion tensor is given by

$$\hat{D}_\mu F = \partial_\mu F + p_\lambda \hat{\Gamma}^\lambda{}_{\mu\nu} \frac{\partial F}{\partial p_\nu} \tag{5}$$

where F is any function. The Poisson-Dirac brackets are expressed as

$$\{F, G\} = \hat{D}_\mu F \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \hat{D}_\mu G - 2 p_\lambda T^\lambda{}_{\mu\nu} \frac{\partial F}{\partial p_\mu} \frac{\partial G}{\partial p_\nu} \tag{6}$$

and the fundamental brackets are

$$\{q^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{p_\mu, p_\nu\} = 2 p_\lambda T^\lambda{}_{\mu\nu}. \tag{7}$$

In the phase space, the geodesic equations read

$$\dot{q}^\alpha = \{q^\alpha, H\} = p^\alpha, \qquad \dot{p}^\alpha = \{p^\alpha, H\} = \hat{\Gamma}^\lambda{}_{\mu\nu} p^\mu p^\nu. \tag{8}$$

The complete integrability of these equations are secured with the existence of the Lax pair equations: [6]

$$\dot{L} = \{L, H\} = [L, A]. \tag{9}$$

Referring to [2], it will be assumed that L is a homogeneous first order polynomial in momenta $L^\alpha{}_\beta = L^\alpha{}_\beta{}^\gamma p_\gamma$. The second matrix A is also of the same form with respect to the momenta $A^\alpha{}_\beta = A^\alpha{}_\beta{}^\gamma p_\gamma$. These third rank objects are referred as the Lax tensor and the Lax connection, respectively. After these preliminaries the brackets $\{L, H\}$ are evaluated for the time derivative of the Lax matrix

$$\dot{L}^\alpha{}_\beta = (L^\alpha{}_\beta{}^{(\mu}{}_{,\gamma} g^{\nu)\gamma} + L^\alpha{}_\beta{}^\gamma \hat{\Gamma}^{(\mu}{}_\gamma{}^{\nu)}) p_\mu p_\nu. \tag{10}$$

The right hand side of (9) can be written as

$$(L^\alpha{}_\gamma{}^\mu A^\gamma{}_\beta{}^\nu - A^\alpha{}_\gamma{}^\mu L^\gamma{}_\beta{}^\nu) p_\mu p_\nu. \tag{11}$$

Comparing the right hand side of (9) with the left hand side, it is now possible to make the following identification:

$$A^\alpha{}_\beta{}^\gamma = \hat{\Gamma}^\alpha{}_\beta{}^\gamma = \Gamma^\alpha{}_\beta{}^\gamma + T^\alpha{}_\beta{}^\gamma. \tag{12}$$

In general, $B^\alpha{}_{\beta\gamma}$ in (1) is an arbitrary third rank tensor, unless some symmetry properties are imposed on it. However, assuming it to be completely antisymmetric, it can be identified with a geometrical object as Cartan's torsion tensor. Furthermore, with such an identification the Lax tensor equations reduce to a compact form:

$$L_{\alpha\beta(\gamma;\delta)} = 0 \tag{13}$$

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where, the semicolon denotes the covariant differentiation with respect to the affine connection.

Particularly, if $B_{\alpha\beta\gamma} = L_{\alpha\beta\gamma}$, then the right hand side of (1) vanishes. In n-dimensional Euclidean or Minkowskian spacetimes they admit a solution of the form:

$$L_{\alpha\beta\gamma} = F_{\alpha\beta\gamma\kappa}x^{\kappa} + C_{\alpha\beta\gamma}. \tag{14}$$

Here the tensor $F_{\alpha\beta\gamma\kappa}$ satisfies $F_{\alpha\beta\gamma\kappa} = -F_{\beta\alpha\gamma\kappa} = -F_{\alpha\beta\kappa\gamma}$, and $C_{\alpha\beta\gamma}$ is a constant tensor. If $B_{\alpha\beta\gamma}$ is completely antisymmetric, then with $C_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$, the equations (1) and (13) are equivalent.

3   The Lax tensor equations on three dimensional spacetimes

Specifically, the Lax tensor can be split into completely symmetric and antisymmetric parts $L_{\alpha\beta\gamma} = S_{\alpha\beta\gamma} + R_{\alpha\beta\gamma}$. For a more detailed analysis we are confined to three dimensions, for the following reasons: First, the antisymmetric part of the Lax tensor becomes proportional to $\epsilon_{\alpha\beta\gamma}$. Then equation (13) reduces to a simple form

$$S_{\alpha\beta\gamma;\mu} = 0 \tag{15}$$

for the symmetric part when $n < 4$, while for $n < 3$ the introduction of a completely antisymmetric third rank tensor is not possible, to identify it with torsion.

In view of (15), $S_{\alpha\beta\gamma}$ is a covariantly constant tensor. Spacetimes admitting such tensors are analyzed in the context of recurrent tensors when $T_{\alpha\beta}{}^{\gamma} = 0$ [16]. The integrability condition for (15) can be expressed as:

$$S_{\mu[\alpha\beta}R^{\mu}{}_{\gamma]\rho\sigma} + 2\,\partial_{[\rho}(T_{\sigma][\alpha}{}^{\mu}S_{\beta\gamma]\mu}) = 0. \tag{16}$$

For simplicity in the notation and when no confusion is possible, [ ] either denotes antisymmetrization, or cyclic permutations. After a detailed analysis of the above consistency condition for an arbitrary diagonal metric with $T_{\alpha\beta}{}^{\gamma} = 0$, we have found that all six surviving components of the Riemann tensor are equal and the surviving Lax tensor components are related as: $S_{111} = S_{333} = 2\,S_{123}$, $S_{222} = -2S_{123}$. However, these restrictions turned out to be very severe on the manifold, yielding the metric components to be constants. Therefore, we can state that the Lax equations are integrable if and only if the manifold is flat.

In the following we will present some examples for the solutions of Lax equations in three dimensions.

3.1 Examples

3.1.1 The three dimensional Rindler system

One of the many possible generalizations of a two dimensional Rindler system to three dimensions can be obtained by assuming the coordinates as

$$x = r \cosh \tau \cos \theta, \qquad y = r \cosh \tau \sin \theta, \qquad t = r \sinh \tau \qquad (17)$$

with $0 < r < \infty, \quad 0 \le \theta < 2\pi, \quad -\infty < \tau < \infty$. The associated metric is

$$ds^2 = dr^2 + r^2 \cosh^2 \tau \, d\theta^2 - r^2 \, d\tau^2. \qquad (18)$$

The $\theta = 0$ hypersurface defines the well-known two dimensional Rindler system [18]. The symmetric Lax tensor has ten components in three dimensions and without torsion the solutions of (15) are found to be

$$
\begin{aligned}
S_{111} &= 3\exp(\tau) - 3\exp(-\tau) + \exp(3\tau) - \exp(-3\tau), \\
S_{113} &= r\left(\exp(\tau) + \exp(-\tau) + \exp(3\tau) + \exp(-3\tau)\right), \\
S_{133} &= r^2\left(-\exp(\tau) + \exp(-\tau) + \exp(3\tau) - \exp(-3\tau)\right), \\
S_{333} &= r^3\left(-3\exp(\tau) - 3\exp(-\tau) + \exp(3\tau) + \exp(-3\tau)\right), \qquad (19) \\
S_{122} &= \tfrac{1}{2} r^2 \exp(-3\tau)\left(\exp(6\tau) + \exp(4\tau) - \exp(2\tau) - 1\right), \\
S_{223} &= \tfrac{1}{2} r^3 \exp(-3\tau)\left(\exp(6\tau) + 3\exp(4\tau) + 3\exp(2\tau) + 1\right), \\
S_{112} &= 0, \qquad S_{123} = 0, \qquad S_{233} = 0, \qquad S_{222} = 0.
\end{aligned}
$$

There are several relations between the Lax tensors and second rank Killing tensors. A second rank Killing tensor satisfies

$$D_{[\mu} K_{\alpha\beta]} = 0. \qquad (20)$$

When the spacetime is flat a Lax tensor can be constructed as: [8]

$$L_{\alpha\beta\gamma} = D_\alpha K_{\beta\gamma} - D_\beta K_{\alpha\gamma}. \qquad (21)$$

A solution to (20), for this metric is found as

$$K_{11} = 1, \qquad K_{22} = \tau, \qquad K_{33} = (1 + r^2)\, r^2 \cosh^2 \tau. \qquad (22)$$

When a second rank Killing tensor is non-degenerate, it can be considered as a metric itself, defining a "dual" spacetime [7]. Although, three dimensional Rindler system defines a flat spacetime, its dual spacetime is curved, with a curvature scalar $R = \frac{-2\,(8r^2+9)}{(r^2+1)^2}$. In view of (21) and (22) we obtain

$$L_{122} = 3r^3 \cosh^2 \tau, \qquad L_{133} = -3r^3 \qquad (23)$$

as the two surviving components of the Lax tensor. Further relations between the Killing tensor and the Lax tensor can be established as $k_{\mu\nu} = S_{\mu\alpha\beta}S_\nu{}^{\alpha\beta}$. Such a Killing tensor is trivial [17]. Its surviving components are:

$$k_{11} = 4\exp(-2\tau)(\exp(4\tau) - 10\exp(2\tau) + 1),$$
$$k_{13} = 4r\,\exp(-2\tau)(\exp(2\tau) + 1)(\exp(\tau) + 1)(\exp(\tau) - 1),$$
$$k_{22} = -8r^2\,\exp(-2\tau)(\exp(2\tau) + 1)^2,$$
$$k_{33} = 4r^2\,\exp(-2\tau)(\exp(4\tau) + 10\exp(2\tau) + 1).$$

$$(24)$$

The dual spacetime associated to this Killing tensor is flat.

3.1.2 Ellipsoidal coordinates system

Separable coordinate systems in three-dimensional Minkowski space label confocal surfaces of order two [18]. In the following coordinates will be denoted by μ, ν, and ρ, defined on the intervals $-\infty < \nu < 0 < \rho < 1 < a < \mu < \infty$, where $a \in R$. The metric corresponding of the Ellipsoidal coordinates systems, falls into this class

$$g_{ij} = \frac{1}{4}diag\left[\frac{(\mu - \nu)(\mu - \rho)}{\mu(\mu - 1)(\mu - a)}, \frac{(\nu - \mu)(\nu - \rho)}{\nu(\nu - 1)(\nu - a)}, \frac{(\rho - \mu)(\rho - \nu)}{\rho(\rho - 1)(\rho - a)}\right]. \quad (25)$$

Non-degenerate Killing metrics corresponding to the Ellipsoidal coordinate system are immediately calculated as:

$$k_{ij} = \frac{1}{\mu\nu\rho}diag\left[\frac{(\mu - \nu)(\mu - \rho)}{(\mu - a)(\mu - 1)}, \frac{(-\mu + \nu)(\nu - \rho)}{(\rho - 1)(\nu - a)}, \frac{(-\mu + \rho)(-\nu + \rho)}{(\rho - 1)(\rho - a)}\right]. \quad (26)$$

For $L_{\alpha\beta\gamma} = -L_{\beta\alpha\gamma}$ solutions to (13) are found as:

$$L_{121} = (\mu - \rho)(\mu + \nu - 2a)/\left[2\mu(\mu - 1)(a - \rho)(\nu - a)(\mu + a)^2\right],$$
$$L_{122} = (\nu - \rho)(\mu + \nu - 2a)/\left[2\nu(\nu - 1)(a - \rho)(\nu - a)^2(a - \mu)\right],$$
$$L_{131} = (\nu - \mu)(2a - \mu - \rho)/\left[2\mu(\mu - 1)(a - \rho)(\nu - a)(a - \mu)^2\right],$$
$$L_{133} = (\rho - \nu)(2a - \mu - \rho)/\left[2\rho(\rho - 1)(a - \rho)^2(\nu - a)(a - \mu)\right],$$
$$L_{232} = (\mu - \nu)(\nu + \rho - 2a)/\left[2\nu(a - \rho)(\nu - a)^2(a - \mu)\right],$$
$$L_{233} = (\mu - \rho)(\nu + \rho - 2a)/\left[2\rho(\rho - 1)(a - \rho)^2(\nu - a)(a - \mu)\right],$$
$$L_{231} = 0, \qquad L_{123} = 0, \qquad L_{132} = 0.$$

$$(27)$$

3.1.3 The three-particle open Toda lattice

The dynamics of the three-particle open Toda Lattice can be formulated through a purely kinetic Hamiltonian

$$H = \frac{1}{2}[(1 + 2a_1{}^2)p_1{}^2 + p_2{}^2 + (1 + 2a_2{}^2)p_3{}^2]. \quad (28)$$

In view of (4) the components of the diagonal metric are found to be

$$g_{11} = (1 + 2a_1{}^2)^{-1}, \qquad g_{22} = 1, \qquad g_{33} = (1 + 2a_2{}^2)^{-1}. \qquad (29)$$

At this point we refer to Sec. 2 and relax the totally antisymmetric condition on the torsion tensor, but consider it in its most general form, which is $T_{\alpha\beta}{}^\gamma = -T_{\beta\alpha}{}^\gamma$, still keeping the metricity condition. Even with this relaxation the autoparalles are retained on the manifold. Now, the affine connection differs from the Christoffel symbols by the contorsion tensor $K_{\alpha\beta}{}^\gamma = -K_{\beta\alpha}{}^\gamma$ as: [19]

$$\hat{\Gamma}_{\alpha\beta}{}^\gamma = \Gamma_{\alpha\beta}{}^\gamma + K_{\alpha\beta}{}^\gamma \qquad (30)$$

whose relation to the torsion tensor is defined through $K_{\alpha\beta}{}^\gamma := T_{\alpha\beta}{}^\gamma - T_\beta{}^\gamma{}_\alpha + T^\gamma{}_{\alpha\beta}$. The surviving components of $L_{\alpha\beta}{}^\gamma (= L_{\beta\alpha}{}^\gamma)$ are found

$$\begin{aligned} L_{11}{}^1 &= g_{11}, & L_{12}{}^1 &= a_1\sqrt{g_{11}}, & L_{22}{}^2 &= 1, \\ L_{23}{}^3 &= a_2\sqrt{g_{33}}, & L_{33}{}^3 &= g_{33}. \end{aligned} \qquad (31)$$

We give the surviving components of the contorsion tensor as:

$$K_{12}{}^1 = a_1\sqrt{g_{11}} - 2a_1^2 g_{11}, \qquad K_{23}{}^3 = a_2\sqrt{g_{33}} - 2a_2^2 g_{33}. \qquad (32)$$

The tensorial Lax equation (1) is satisfied when $L_{\alpha\beta}{}^\gamma$ is as in (31) and $B_{\alpha\beta}{}^\gamma = K_{\alpha\beta}{}^\gamma$ is as above. Therefore, for this particular example $B_{\alpha\beta}{}^\gamma$ can be interpreted as the contorsion tensor.

4 Conclusion

In this paper, we generalized the Lax tensor equations introduced by Rosquist, by appropriately defining the Poisson brackets in the presence of torsion. This way, otherwise arbitrary tensors of these equations can be identified with concrete geometrical objects, such as the torsion or the contorsion tensor, when some relevant symmetry properties are imposed on them. The form of the equations are considerably simplified, when $B^\alpha{}_{\beta\gamma}$ is completely antisymmetric.

We have also found the conditions when the Lax equation on a three dimensional manifold admit solutions. We analyzed separable coordinates and the three-particle open Toda lattice, in detail.

As was pointed in [17] Killing tensors can be trivial or non-trivial. A similar characterization arises when we investigate the solutions of the Lax tensor equations. If the Lax tensors satisfy $g_{\mu\nu} = L_{\mu\alpha\beta}L_\nu{}^{\alpha\beta}$, then they are non-trivial tensors.

Further intriguing problems are to investigate the existence of the Lax tensors, when the manifold admits Runge-Lenz symmetry, or to find Lax tensors for superintegrable systems. These problems are currently under investigation [20].

Acknowledgments

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